ON DISCRETE-TIME PROCESSES
IN A PACKETIZED COMMUNICATION SYSTEM

BY

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Abstract

In the analysis of a packetized communication system such as a slotted ALOHA, one needs to deal with discrete-time point processes and related queues. We examine how Poisson processes, M/M/1 queue, M/G/∞, etc., should be translated in their discrete-time versions. We then give a new definition of processor-sharing and its properties. Correlation functions of input and output processes in an M/G/∞ are also studied.

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I. INTRODUCTION

In the analysis of a packetized communication system, one needs to deal with discrete-time point processes. One may quickly conclude that most results and formulas known for continuous-time processes can be translated into their discrete time versions in obvious manner. Unfortunately, this turns out not to be the case. The main problem lies in that a Poisson process has two discrete time analogs, one is a Bernoulli sequence, and the other is a Poisson sequence. A discrete time version of an M/M/1 system holds for a Bernoulli arrival sequence, whereas an M/G/∞ can be translated only for a Poisson sequence as will be discussed in Sections II and III. Important results obtained for an M/G/1 system under processor sharing (PS) cannot be interpreted for the round-robin (RR) scheduling, despite the fact the notion of the PS scheduling was originally derived from RR. Thus we introduce a new notion of processor sharing.

This note is therefore intended to examine some fundamentals issues of discrete time point processes and related queueing systems. Applications of these results to modeling and performance analysis of a slotted ALOHA SYSTEM will be discussed in subsequent reports.

II. A POISSON SEQUENCE WITH RANDOM DELAY: SYSTEM M/G/∞

Definition 2.1: A discrete-time point process X_k is called a Poisson
sequence if $\{X_k : k=0,1,2,\ldots\}$ are i.i.d. with Poisson
distribution of constant mean λ.

Consider a point process $X(t)$, and let us define X_k as the number
of points (arrivals) in the interval $(k-1)T \leq t < kT$. Clearly, if $X(t)$ is
a Poisson process with rate μ, then X_k is a Poisson sequence with mean
$\lambda = \mu T$. For example, the number of messages arriving into a slotted ALOHA
channel from a large number of terminals can be accurately characterized
by a Poisson sequence.

Random Delay

Suppose that each element (e.g., message or job) of a discrete-time
point process is passed through a "random delay" (RD) as depicted in Figure
2.1. Each element (message) stays in one of the infinite boxes for a
random amount of time d. The delays of the individual messages are chosen
independently, but from a common distribution $\{f(d)\}$, $\sum\limits_{d=0}^{\infty} f(d) = 1$. Backlogged
messages in a slotted ALOHA channel with random access scheme, for example,
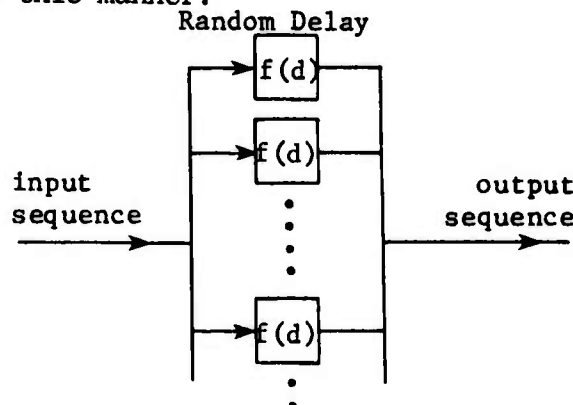can be modelled in this manner.

Figure 2.1

The notion of random delay is equivalent to a service station which has infinite servers in parallel. In this case f(d) corresponds to service time distribution. The notion of infinite servers is applicable even to a finite set of parallel servers, provided the number is sufficient that no queue is formed. For example, a collection of user terminals in a finite population model of an ALOHA SYSTEM can be characterized by RD, in which f(d) is the probability of the terminal think time d.

Property 2.1:   For a Poisson arrival sequence with rate λ, the probability that there are n jobs (messages) in the RD at time k is

$$p(n;k) = \frac{\left\{\lambda \sum_{d=0}^{k-1} F^C(d)\right\}^n}{n!} \exp \left\{-\lambda \sum_{d=0}^{k-1} F^C(d)\right\} \qquad (2.1)$$

where $F^C(d)$ is the complementary distribution of delay d:

$$F^C(d) = \text{Prob[delay>d]}$$

$$= \sum_{i>d} f(i) \qquad (2.2)$$

In the limit $k\to\infty$, we have the following equilibrium distribution

$$p(n) = \frac{\rho^n}{n!} e^{-\rho} \qquad (2.3)$$

where

$$\rho = \lambda E[d] \qquad (2.4)$$

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Furthermore, the out (departing) process from RD is also a Poisson sequence with mean λ.

Proof. See Appendix A.

Different Classes of Messages or Terminals

Suppose that there are R different classes of messages (or terminals) with different delay (or think time) distributions $F_r(d) = \sum_{i=0}^{d} f_r(i)$ $r = 1, 2, \ldots R$. Let us denote the system state by vector

$$\underline{n} = [n_1, n_2, \ldots, n_R]$$

where n_r is the number of jobs in RD at a given time. If class-r messages arrivals are independent Poisson sequences with rates λ_r, the probability of the system state at time k is

$$p(n; k) = \prod_{r=1}^{R} p_r(n_r; k) \tag{2.5}$$

where $p_r(n_r; k)$ takes the form of (2.1). In the limit $k \to \infty$

$$p(\underline{n}) = \prod_{j=1}^{R} \frac{\rho_r^{n_r}}{n_r!} e^{-\rho_r} \tag{2.6}$$

The corresponding probability generating function (p.g.f.) is given by

$$Q(\underline{z}) = E\left[\prod_{r=1}^{R} z_r^{n_r} \right] = \exp\left\{ -\rho + \sum_{r=1}^{R} \rho_r z_r \right\} \tag{2.7}$$

where

$$\rho = \sum_{r=1}^{R} \rho_r = \sum_{r=1}^{R} \lambda_r E[d_r] \tag{2.8}$$

Thus, the p.g.f. of the marginal distribution of the total number of messages in RD is

$$Q(z) = \exp\{-\rho + \rho z\} \qquad (2.9)$$

which yields the distribution of $\quad n = \sum_{r=1}^{k} n_r :$

$$p(n) = \frac{\rho^n}{n!} e^{-\rho} \qquad (2.10)$$

III. A BERNOULLI SEQUENCE AND GEOMETRIC DISTRIBUTION OF MESSAGE LENGTH: SYSTEM M/M/1

Definition 3.1:   A discrete time 0-1 valued process X_k is called a *Bernoulli sequence* with parameter λ if $\{X_k; k=0,1,2,...\}$ are i.i.d. with distribution $P[X_k=1]=\lambda$, and $P[X_k=0]=1-\lambda$, for all k.

Let T_i be the number of zero runs (which hereafter we call "run-length") between the $(i-1)^{st}$ one and the ith one in $\{X_k\}$. Then $\{T_i\}$ are i.i.d. with geometric distribution.

$$P[T_i=t] = \lambda(1-\lambda)^t \qquad \ell=0,1,2,... \qquad (3.1)$$

Thus the average run-length is

$$E[T_i] = \frac{1-\lambda}{\lambda} \qquad (3.2)$$

For example, the sequence of messages from a user terminal is often modelled as a Bernoulli sequence, and the variable T is often called the "thinking time". In a discrete-time queueing system, the Bernoulli process plays a role equivalent to that of a Poisson process in a continuous-time queueing system.

Definition 3.2: A discrete-time queueing system is denoted by M/M/1 if its message (job) arrival sequence is a Bernoulli sequence, the message lengths (job service times) are i.i.d. with a geometric distribution, and there is only one channel (server).

<u>Definition 3.3</u>: A queueing (scheduling) discipline is said to be a *work-conserving* discipline [8] if (1) the lengths of each message are not affected by the queue discipline; (2) the queue discipline cannot take advantage of possible knowledge about message arrival times or message lengths; and (3) the channel should not be idle if there are some messages to be sent.

Because of the so-called memoryless property of geometric distribution, the state of the system M/M/1 under a work-conserving discipline can be defined simply by n, the number of messages in the system (either being transmitted over the channel or waiting in the queue). Suppose that the message generating probability is dependent on the system state, and we denote it by $\lambda(n)$ [message/slot time]. Similarly, assume that service rate (or channel speed) is dependent on the system state and denoted by $C(n)$ [packets/slot time]. If the number of packet length L in a message is geometrically distributed with parameter r (i.e., the average message has the length $\bar{L} = \frac{1-r}{r}$ [packets])

$$P[L=\ell] = r(1-r)^{\ell} \qquad \ell = 0,1,2,\ldots \qquad (3.3)$$

we obtain the following result by solving a discrete time version of the birth-and-death process:

<u>Property 3.1</u>: The equilibrium state distribution of n in the M/M/1 under a work-conserving discipline, if it exists, is given by

$$p(n) = p(0)\Lambda(n)D(n)\bar{L}^{n} \qquad (3.4)$$

where

$$\Lambda(n) = \prod_{i=1}^{n} \lambda(i-1) \tag{3.5}$$

$$D(n) = \prod_{i=1}^{n} \frac{1}{C(i)} \tag{3.6}$$

Note the existence of the equilibrium distribution (i.e., the condition for system stability) is equivalent to the condition that the p.g.f.

$$Q(z) = p(0) \sum_{n=0}^{\infty} \Lambda(n)D(n)(Lz)^n \tag{3.7}$$

is analytic inside the unit disc $|z|=1$.

The unknown constant is given by setting $Q(1)=1$:

$$p(0)^{-1} = \sum_{n=0}^{\infty} \Lambda(n)D(n)L^n \tag{3.8}$$

From the above result we can also show

Property 3.2:   The output (departure) process with time "reversed" is a
Bernoulli process with rate $\lambda(n)$, where n is the "current"
state of the system. Thus if the arrival is a homogeneous
Bernoulli process with rate λ, so is the output sequence.

An exact analogy between continuous-time and discrete-time queueing
systems breaks down, however, because Bernoulli processes do not possess the
"reproducing" property under addition. Consider a time interval of arbitrary
length [time slots] and consider m independent Bernoulli sequences. The
number of arrivals n_k from the kth stream during this interval is the binomial

distribution with mean λ_k

$$p[n_k] = \binom{N}{n_k} \lambda_k^{n_k} (1-\lambda_k)^{N-n_k} \tag{3.9}$$

which has the p.g.f.

$$G_k(z) = \{1-\lambda_k(1-z)\}^N \tag{3.10}$$

Thus the total number of arrivals from all to m sources has the p.g.f.

$$G^{(m)}(z) = \prod_{k=1}^{m} G_k(z) = \prod_{k=1}^{m} \{1-\lambda_k(1-z)\}^N \tag{3.11}$$

If in particular $\lambda_k = \dfrac{\lambda}{m}$ for all k=1,2,...,m

$$G^{(m)}(z) = \left\{1 - \frac{\lambda}{m}(1-z)\right\}^{mN} \tag{3.12}$$

In the limit

$$\lim_{m \to \infty} G^{(m)}(z) = e^{-\lambda(1-z)N} \tag{3.13}$$

Thus the superposition of many independent Bernoulli sequences with same rates converges in the limit to a Poisson sequence.

IV. PROCESSOR-SHARING AND GENERAL MESSAGE LENGTH DISTRIBUTION

Definition 4.1: We define processor-sharing in a discrete time queueing system as a dispatching algorithm which selects, in every slot time, one of n messages in the system randomly and equally likely.

For example, we can approximate a slotted ALOHA channel by a processor shared server with queue-dependent rate $C(n)$. (How to determine the function $C(n)$ for a slotted ALOHA channel will be discussed in details in a forthcoming report.) Note that the notion of processor-sharing in a discrete-time system is different from the so-called round-robin (RR) scheduling. When the time quantum or slot time approaches zero, the notions of RR and PS defined above are both reduced to the familiar notion of PS in continuous-time queueing theory. We can establish the following important result on PS. The corresponding result in continuous-time queueing system is discussed in many recent articles (e.g., [1,2]).

Property 4.1: Let a message sequence be characterized by a Bernoulli sequence with rate $\lambda(n)$, and let the distribution of the number of packets in a message be a general distribution $F_L(\ell)$ with mean $\overline{L}$ [packets], i.e.,

$$\overline{L} = \sum_{\ell=0}^{\infty} F_L^C(\ell) \tag{4.1}$$

If the queue discipline is PS and processing rate is $C(n)$ [packets/slot time], then the equilibrium distribution of n is given by the formula (3.4):

$$p(n) = p(0)\Lambda(n)D(n)\overline{L}^n \tag{4.2}$$

<u>Proof</u>: Let $\Phi(z)$ be the p.g.f. or z-transform of the message length distribution:

$$\Phi(z) = \sum_{i=0}^{\infty} z^i \{F_L(i) - F_L(i-1)\} \qquad (4.3)$$

which is a discrete analog of the Laplace-Stieltjes transform. If $\Phi(z)$ can be written as a rational function of z:

$$\Phi(z) = \frac{R(z)}{Q(z)} \qquad (4.4)$$

we can obtain the following expansion

$$\Phi(z) = b_0 + \sum_{r=1}^{q} a_0 a_1 \cdots a_{r-1} b_r \prod_{i=1}^{r} \frac{1-\alpha_i}{1-\alpha_i z} \qquad (4.5)$$
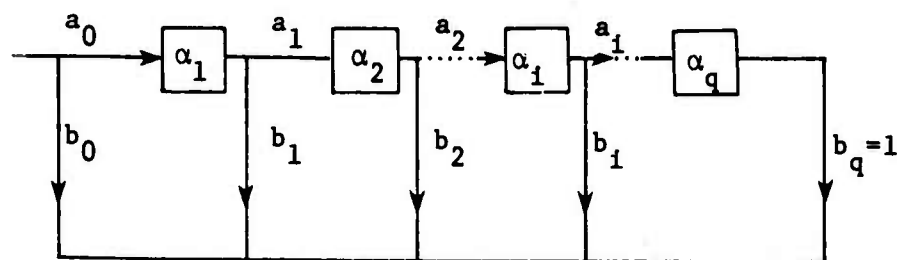

Figure 4.1

where q is the degree of the polynomial $Q(z)$, and $\{\alpha_i^{-1}\}$ are the characteristic roots of $Q(z)=0$. The coefficients $\{a_i\}$ and $\{b_i\}$ satisfy

$$a_i + b_i = 1 \qquad 0\leq i\leq q-1 \qquad (4.6)$$

and

$$b_q = 1 \qquad (4.7)$$

The representation (4.5) is schematically shown in Figure 4.1, which can be viewed as cascaded geometric distributions* with parameters $\alpha_1,\alpha_2,\ldots,\alpha_q$. This expansion is a discrete analog of Cox's representation [3] of a general service time distribution whose Laplace-Stieltjes transform is a rational function of the Laplacian variable s. Appendix B discusses illustrative examples of the expansion (4.5).

We define the system state by the vector $\underline{n}=\{n_j,1\leq j\leq q\}$ where n_j is the number of messages in the system which are currently in the jth (fictitious) geometric server in the representation of Figure 4.1. We define $\underline{n}(j^-)$ as the state obtained by reducing its jth component n_j by one, and $\underline{n}(j^+)$ is obtained by adding unity to the jth component. Similarly the transition {state $\underline{n}(j^+,i^-)$ → state $\underline{n}$} takes place if one of the jth elements moves to the ith component. Let $P[\underline{n};k]$ be the probability that the system is in state $\underline{n}$ at slot time k.

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* Note that this distribution takes a slightly different form from (3.3):
$$P[L_i=\ell] = \alpha_i^{\ell-1}(1-\alpha_i) \qquad \ell=1,2,3,\ldots$$

To calculate $P[\underline{n};k+1]$, we note that the system equation can be written as follows:

$$P[\underline{n};k+1] = P[\underline{n};k]\{1-\lambda(n) - \frac{1}{n}\sum_{i=1}^{q} n_i\overline{\alpha}_i C(n)\} + P[\underline{n}(1^-);k]\lambda(n-1)$$

$$+ \frac{1}{n+1}\sum_{j=1}^{q} P[\underline{n}(j^+);k](n_j+1)\overline{\alpha}_j b_j C(n+1)$$

$$+ \frac{1}{n}\sum_{j=2}^{q} P[\underline{n}(j-1^+,j^-);k](n_{j-1}+1)\overline{\alpha}_{j-1}a_{j-1}C(n) \qquad (4.8)$$

where
$$\overline{\alpha}_j = 1 - \alpha_j \qquad (4.9)$$

The steady state distribution, if it ever exists, must satisfy the balance equation (4.8) in which we set $\lim_{k\to\infty}P[\underline{n};k+1]=\lim_{k\to\infty}P[\underline{n};k]=P[\underline{n}]$. Then after a rather extensive manipulation we can show that the following recurrence relation is a sufficient condition for the global balance equation.

$$\frac{n_i}{n}\overline{\alpha}_i C(n)P[\underline{n}] = e_i\lambda(n-1)P[\underline{n}(k^-)] \qquad (4.10)$$

for all $i=1,2,\ldots q$ and for all states $\underline{n}$. The equation (4.10) is a discrete system equivalent of the local balance equation or individual balance equation discussed in [4,5,1,2]. In (4.10) the new quantity e_k is the probability that a message ever reaches the kth (fictitious) service stage:

$$e_k = \prod_{i=1}^{k-1} a_i \qquad (4.11)$$

By using the recurrence equation repeatedly we obtain

$$P[\underline{n}] = P(0)\Lambda(n)D(n)n!\prod_{i=1}^{q} \frac{1}{n_i!}\left(\frac{e_i}{\overline{\alpha}_i}\right)^{n_i} \qquad (4.12)$$

Then the marginal distribution of $n=n_1+n_2+\ldots+n_q$ is given by

$$p(n) = p(0)\Lambda(n)D(n)\overline{L}^n \qquad (4.13)$$

where

$$\bar{L} = \sum_{i=1}^{q} \frac{e_i}{\bar{\alpha}_i} = \sum_{\ell=0}^{\infty} \{1-F_L(\ell)\} \qquad (4.14)$$

The output (departure) process from the PS system has the same property as M/M/1 which was stated as Property 3.3.

Different Classes of Messages

The above result can be generalized to messages with different classes.

Property 4.2: Let R independent message streams be characterized by Bernoulli sequences with rates $f_r\lambda(n)$, where f_r is the fraction of the rth class, $\sum_{r=1}^{R} f_r=1$. Let the message length distributions be $F_r(\ell)$, r=1,2,...,R. If the channel (or server) adopts the PS discipline with rate C(n), the joint distribution, $P[\{n_r\}]$, in equilibrium is given by

$$P[\{n_r\}] = P(0)\Lambda(n)D(n)n! \prod_{r=1}^{R} \frac{1}{n_r!} (\bar{L}_r f_r)^{n_r} \qquad (4.15)$$

where $\bar{L}_r$ [packets] is the average length of the class-r message:

$$\bar{L}_r = \sum_{\ell=0}^{\infty} \{1-F_r(\ell)\} \qquad (4.16)$$

Hence, the distribution of the total number of messages in the system is

$$P(n) = P(0)\Lambda(n)D(n)\bar{L}^n \qquad (4.17)$$

where

$$\overline{L} = \sum_{r=1}^{R} f_r \overline{L}_r \tag{4.18}$$

V. CORRELATION FUNCTIONS

In this section we discuss some statistical properties of discrete time processes observed in a system like slotted ALOHA. One of the crucial assumptions usually made in the analysis of such system is the Poisson sequence approximation of the total traffic into a channel. As the first step toward a more accurate characterization, correlation functions of the related processes are examined.

Let $\{X_k\}$ be a stationary discrete time point process and be passed into the RD (random delay device) and we denote the output sequence by $\{Y_k\}$. We define autocorrelation and cross-correlation functions as in the usual way:

$$R_{XX}(k,\ell) \triangleq E[(X_k-\overline{X})(X_\ell-\overline{X})] \tag{5.1}$$

$$R_{XY}(k,\ell) \triangleq E[(X_k-\overline{X})(Y_\ell-\overline{Y})] \tag{5.2}$$

etc.

Property 5.1: For a wide-sense stationary sequence $\{X_k\}$, we have

$$R_{XY}(k,\ell) = \sum_{\ell'} R_{XX}(k,\ell')f_{\ell-\ell'} \tag{5.3}$$

and

$$R_{YY}(k,\ell) = \sum_{k'}\sum_{\ell'} f_{k-k'}\{R_{XX}(k'\ell')-\overline{X}\delta_{k',\ell'}\}f_{\ell-\ell'}$$

$$- \overline{X}\delta_{k,\ell}\{1- \sum_{k'} f_{k-k'}f_{\ell-k'}\} \tag{5.4}$$

where $\{f_k\}$ is the probability that the delay length is k [time slots], and $\delta_{k,\ell}$ is Kronecker's delta.

Proof: See Appendix C.

Equation (5.3) has a resemblance to the well known formula for the cross-correlation between a linear filter input and output, in which the impulse response of the filter is $\{f_k\}$. Equation (5.4) is different, however, from the corresponding expression for the linear filter output, which does not possess the second term of (5.4).

For example, if the input is a Poisson sequence with rate λ, then $R_{XX}(k,\ell')=\lambda\delta_{k\ell'}$. Therefore,

$$R_{XY}(k,\ell) = \lambda f_{\ell-k} \tag{5.5}$$

and

$$R_{YY}(k,\ell) = \lambda \sum_{k'} f_{k-k'} f_{\ell-k'} - \lambda \sum_{k'} f_{k-k'} f_{\ell-k'} + \lambda\delta_{k,\ell}$$

$$= \lambda\delta_{k,\ell} \tag{5.6}$$

which is consistent with the earlier result that $\{Y_k\}$ is also a Poisson sequence.

As the second example, consider the case where X_k is a Bernoulli sequence with rate λ. Then $\overline{X} = \lambda$ and

$$R_{XX}(k,\ell) = \lambda(1-\lambda)\delta_{k,\ell} \tag{5.7}$$

Thus

$$R_{YY}(k,\ell) = -\lambda^2 \sum_{k'} f_{k-k'} f_{\ell-k'} + \lambda\delta_{k,\ell} \tag{5.8}$$

which is sufficient to conclude that Y_k is neither Bernoulli nor Poisson.

APPENDIX A

Proof of Property 2.1

The following proof is essentially a discrete analog of the result known for $M/G/\infty$ [7]. Consider a message which has arrived by slot time k. The uniformity of Poisson process indicates that the arrival time j is uniformly distributed over k slots. The message will be still in RD at slot time k ($\geq$ j) with prob $P[delay \geq k-j+1] = 1 - F(k-j) = F^C(k-j)$. The probability that a message is still in RD at slot k given the condition that the job has arrived by that slot is therefore

$$p = \frac{1}{k} \sum_{j=1}^{k} F^C(k-j) = \frac{1}{k} \sum_{d=0}^{k-1} F^C(d) \qquad (A-1)$$

Clearly

$$\lim_{k \to \infty} p = \frac{E[d]}{k} \qquad (A-2)$$

Suppose that n messages have arrived by slot time k. Since their delay times in the RD are i.i.d., the probability that i messages are still in RD at time k is given by

$$\binom{n}{i} p^i (1-p)^{n-i} \qquad 0 \leq i \leq n \qquad (A-3)$$

For a Poisson arrival sequence with rate λ, the number n is Poisson distributed with mean λk:

$$\frac{(\lambda k)^n}{n!} e^{-\lambda k} \qquad (A-4)$$

Thus the probability that there are i messages in RD at time k is:

$$p_i(k) = \sum_{n=i}^{\infty} \frac{(\lambda k)^n}{n!} e^{-\lambda k} \cdot \binom{n}{i} p^i (1-p)^{n-i} = \frac{(\lambda k p)^i}{i!} e^{-\lambda k p} \qquad (A-5)$$

By substituting (A-1) into (A-5), we obtain (2.1).

With p defined by (A-1), 1-p represents the probability that an arbitrary message which has arrived by slot k has departed already. Therefore, the total number of messages j that have departed by the end of slot k is obtained by substituting 1-p for p in (A-5):

$$q_j(k) = \frac{\{\lambda k(1-p)\}^j}{j!} e^{-\lambda k(1-p)} \qquad (A-6)$$

which is again a Poisson distribution but with mean $\lambda \sum_{d=0}^{k-1} F(d)$. We can show that the number leaving RD at the end of (k-1)st slot is independent of the number of messages left at slot time k. This independence of the past departure process and current state of the system is important. It follows that the number of departures in different slots are statistically independent. Thus, we can show that the number of departure in slot k is Poisson distributed with mean $\lambda F(k-1)$. In the limit $\lim_{k\to\infty} \lambda F(k-1) = \lambda$.

APPENDIX B

On the Expansion of (4.5)

The geometric stage representation of a general discrete distribution involves the formal use of complex transition probabilities, since the characteristic roots α_i^{-1} of $\Phi(z)$ can take on complex values. Of course we can choose a different type of expansion in which all parameters are real. This involves in general serial-parallel combination of geometric distributions and will not be given in such compact form as the expansion (4.5). Note also that we do not really need to find expansion parameters $\{\alpha_i\}$ $\{a_i\}$ and $\{b_i\}$. The final result (4.13) does not include these parameters. In this sense the representation (4.5) is analogous to the Fourier series expansion or general orthogonal expansion used in signal analysis.

To give the reader some feel of the representation (4.5), we will show illustrative examples in which expansion coefficients are easily found.

(1) The distribution

$$f(k) = \binom{k+q-1}{q-1} (1-\alpha)^q \alpha^k \qquad k=0,1,2,\ldots \qquad \text{(B-1)}$$

is a variant of the negative binomial or Pascal distribution. Its p.g.f. is

$$\Phi(z) = \sum_{k=0}^{\infty} f(k) z^k = \left(\frac{1-\alpha}{1-\alpha z}\right)^q \qquad \text{(B-2)}$$

Thus the expansion (4.5) is simply obtained by setting all a_i's to be unity and $\alpha_1 = \alpha_2 = \ldots = \alpha_q = \alpha$.

(2) The Poisson distribution

$$f(k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad k \geq 0 \qquad \text{(B-3)}$$

has the corresponding p.g.f.

$$\Phi(z) = e^{\lambda(z-1)} \qquad\qquad (B-4)$$

This is clearly not a rational function of z. If we allow, however, the formal passages to the limit we can represent $\Phi(z)$ as

$$\Phi(z) = \lim_{q\to\infty} \left(\frac{1 - \dfrac{\lambda}{q}}{1 - \dfrac{\lambda z}{q}} \right)^q \qquad\qquad (B-5)$$

See [6] for an example in which probability parameter α_i's are complex.

APPENDIX C

Derivation of Property 5.1

We write the output process Y_ℓ as

$$Y_\ell = \sum_{j=0}^{\ell} z_\ell(j) \tag{C-1}$$

where $z_\ell(j)$ is a contribution from X_j to the value Y_ℓ, i.e.,

$$z_\ell(j) = n \quad \text{with} \quad \binom{X_j}{n} f_{\ell-j}^n (1-f_{\ell-j})^{X_j-n} \quad 0 \leq n \leq X_j \tag{C-2}$$

Then

$$E[X_k Y_\ell] = \sum_{j=0}^{\ell} E[X_k z_\ell(j)]$$

$$= \sum_{j=0}^{\ell} \sum_{n=0}^{X_j} n X_k \binom{X_j}{n} f_{\ell-j}^n (1-f_{\ell-j})^{X_j-n} P[X_k, X_j]$$

$$= \sum_{j=0}^{\ell} X_k X_j f_{\ell-j} \{f_{\ell-j} + (1-f_{\ell-j})\}^{X_j-1} P[X_k, X_j]$$

$$= \sum_{j=0}^{\ell} \{R_{XX}(k,j) + \overline{X}^2\} f_{\ell-j} \tag{C-3}$$

Therefore

$$R_{XY}(k,\ell) = E[X_k Y_\ell] - \overline{X}\overline{Y} = \sum_{j=0}^{\ell} R_{XX}(k,j) f_{\ell-j} \tag{C-4}$$

which proves (5.4).

From (C-1) we write for $k \neq \ell$

$$E[Y_k Y_\ell] = E\left[\sum_{i=0}^{k} z_k(i) \sum_{j=0}^{\ell} z_\ell(j)\right]$$

$$= \sum_{\substack{i \ j \\ i \neq j}} \sum f_{k-i} E[X_i X_j] f_{\ell-j} + \sum_i \sum_m \sum_n mn \, E\left[\frac{X_i!}{m!n!(X_i-m-n)!} f_{k-i}^m f_{\ell-i}^n (1-f_{k-i}-f_{\ell-i})^{X_i-m-n}\right]$$

$$= \sum_{\substack{i \ j \\ (i \neq j)}} \sum f_{k-i} E[X_i X_j] f_{\ell-j} + \sum_i E[X_i(X_i-1) f_{k-i} f_{\ell-i}] \qquad (C-5)$$

Hence for $k \neq \ell$

$$R_{YY}(k,\ell) = \sum_i \sum_j f_{k-i} R_{XX}(i,j) f_{\ell-j} - \overline{X} \sum_i f_{k-i} f_{\ell-i} \qquad (C-6)$$

For $k = \ell$, (C-5) should be modified as

$$E[Y_k^2] = \sum_{\substack{i \ j \\ (i \neq j)}} \sum f_{k-i} E[X_i X_j] f_{k-j} + \sum_i \sum_m m^2 E\left[\binom{X_i}{m_i} f_{k-i}^m (1-f_{k-i})^{X_i-m}\right] \qquad (C-7)$$

Then writing $m^2 = m(m-1)+m$, we have

$$R_{YY}(k,k) = \sum_i \sum_j f_{k-i} R_{XX}(i,j) f_{k-j} - \overline{X} \sum_i f_{k-i}^2 + \overline{X} \sum_i f_{k-i} \qquad (C-8)$$

For sufficiently large k, $\sum_{i=0}^{k} f_{k-i} = 1$, hence we obtain (5.4).

REFERENCES

[1] F. Baskett, K. M. Chandy, R. R. Muntz and F. G. Palacios, "Open, Closed, and Mixed Networks of Queues With Different Classes of Customers," *Journal of the ACM*, Vol. 22, April 1975, pp. 248-260.

[2] M. Reiser and H. Kobayashi, "Queueing Networks With Multiple Closed Chains: Theory and Computational Algorithms," *IBM Journal of Research and Development*, Vol. 19, May 1975, pp. 283-294.

[3] D. R. Cox, "A Use of Complex Probabilities in Theory of Stochastic Processes," *Proc. Conf. Phil. Soc.*, Vol. 51, 1955, pp. 313-319.

[4] P. Whittle, "Equilibrium Distributions for an Open Migration Process," *Journal of Applied Probabilities*, Vol. 5, 1968, pp. 567-571.

[5] K. M. Chandy, "The Analysis and Solutions for General Queueing Networks," *Proc. Sixth Annual Princeton Conference on Information Sciences and Systems*, Princeton, New Jersey, March 1972, pp. 224-228.

[6] H. Kobayashi and M. Reiser, "On Generalization Job Routing Behavior in a Queueing Network Model," *IBM Research Report RC 5252*, February 1975, IBM T. J. Watson Research Center, Yorktown Heights, New York.

[7] N. M. Mirasol, "The Output of an M/G/∞ Queue is Poisson," *Operations Research*, Vol. 11, 1963, pp. 282-284.

[8] R. W. Wolff, "Work-Conserving Priorities," *J. Appl. Prob.*, Vol. 7, 1970, pp. 327-337.